Technical Report: Detailed Proofs of Bounds for Map-Aware Localization

Francesco Montorsi*[1] and Giorgio M. Vitetta†[1]

[1]Department of Information Engineering, University of Modena and Reggio Emilia

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*francesco.montorsi@unimore.it
†giorgio.vitetta@unimore.it

1 Introduction

In this Technical Report the proofs of the *Bayesian Cramer Rao bound* (BCRB), *extended Ziv Zikai bound* (EZZB) and *Weiss Weinstein bound* (WWB) are provided for localization systems endowed with map knowledge. For a description of the notation used in the derivations and for a discussion of the bounds, please refer to [1, Sec. II] and [1, Sec. III], respectively.

2 Derivation of the BFIM for Generic-Shape Uniform Maps

We start the derivation for the BFIM associated to uniform maps introducing some properties for the smoothing function mentioned in [1, Sec. II] which is used to model the map pdf. Then the 1-D BFIM is obtained and later extended to 2-D case, thus proving [1, Eq. (8)].

2.1 Modelling of the Pdf

Let $s(t)$ be a continuous and differentiable function $s : \mathbb{R} \to \mathbb{R}$. Assume that $s(t)$ has the following additional properties: 1) $s(t) \geq 0 \, \forall t \in \mathbb{R}$; 2) $\int s(t)dt = 1$; 3) $\int \frac{\partial s(t)}{\partial t}dt = 0$; 4) $s(t)$ has support $\left[-\frac{1}{2}; +\frac{1}{2}\right]$; 5) $s(0) = 1$.

The function $s(\cdot)$ is then a pdf function (assumptions 1 and 2) and has an associated a-priori FI J_s (assumption 3 is the regularity condition that grants the FI existence) [2]. The assumptions 4 and 5 finally assure that $s(\cdot)$ can be used to model bounded statistical distributions i.e. maps as we defined in [1, Sec. II], eventually with some scaling and translation.

A function $s(\cdot)$ satisfying the conditions above is dubbed in [1] as "smoothing function". Examples of functions that satisfy those hypotheses are:

1. $s(t) = g\left(t + \frac{1}{2}; \delta\right) g\left(-t + \frac{1}{2}; \delta\right)$ where $g(t;\delta) \triangleq \frac{f(t+\delta)}{f(t+\delta)+f(-t+\delta)}$ and $f(t) \triangleq e^{-\frac{1}{t}}\mathrm{u}(t)$;

2. $s(t) = g\left(t + \frac{1}{2}; \delta\right) g\left(-t + \frac{1}{2}; \delta\right)$ where

$$g(t;\delta) \triangleq \begin{cases} 0 & t < -\delta \\ -\frac{t^3}{4\delta^3} + \frac{3t}{4\delta} + \frac{1}{2} & -\delta \leq t \leq +\delta \\ 1 & t > +\delta \end{cases}$$

Note that in the examples above $\delta \in \left(0; \frac{1}{2}\right]$ is a parameter which defines the steepness of the pdf $s(t)$, so that $\lim_{\delta \to 0} g\left(t;\delta\right) = \mathrm{u}(t)$, where $\mathrm{u}(t)$ is the unit step function (which however does not have an associated FI). In the latter example, the associated FI is easy to compute in closed form: $\mathrm{J}_s(\delta) = \frac{9 \ln 3}{4\delta}$.

Finally note that $\tilde{s}(t,a,b) \triangleq \frac{1}{b} s\left(\frac{t-a}{b}\right)$ is still a pdf function and its associated FI is:

$$\tilde{\mathrm{J}}_s \triangleq \mathbb{E}_t\left\{\left(\frac{\partial \ln}{\partial t}\frac{1}{b}s\left(\frac{t-a}{b}\right)\right)^2\right\} = \mathbb{E}_t\left\{\left(\frac{\partial \ln s\,(u)}{\partial u}\frac{1}{b}\right)^2\right\} = \frac{\mathrm{J}_s}{b^2}$$

2.2 Derivation of the BFIM for 1-D Maps

Considering a unidimensional (1-D) scenario, where the agent position to estimate is the scalar x. The support $\mathcal{R} \subset \mathbb{R}$ of a 1-D uniform map can be always represented as the union of N_r disjoint segments spaced by $N_r - 1$ segments where the map pdf $f(x)$ is equal to 0. Let the N_r segments of the support $\mathcal{R}$ be indexed by the odd numbers of the set $\mathcal{N}^o = \{1, 3, ..., 2N_r - 1\}$ and let c_n (w_n) denote the centre (width) of the n-th segment, with $n \in \mathcal{N}^o$. Then the map pdf $f(x)$ associated with this scenario can be expressed as

$$f(x) = \frac{1}{\mathcal{W}_\mathcal{R}}\sum_{n\in\mathcal{N}^o} s\left(\frac{x-c_n}{w_n}\right) = \frac{1}{\mathcal{W}_\mathcal{R}}\sum_{n\in\mathcal{N}^o} w_n \tilde{s}(x, c_n, w_n) \tag{1}$$

where $\mathcal{W}_\mathcal{R} \triangleq \sum_{n\in\mathcal{N}^o} w_n$ and $\tilde{s}(\cdot)$ is the smoothing function as defined in Sec. 2.1.

The a-priori FI associated with $f(x)$ is $\mathrm{J}_x \triangleq \mathbb{E}_x\left\{\left(\frac{\partial \ln f(x)}{\partial x}\right)^2\right\}$ [2]. To simplify the expression we consider that a) for each value of x there is only one rectangle at most for which $\left(\frac{\partial \ln f(x)}{\partial x}\right)^2 \neq 0$, and b) varying x over $\mathbb{R}$, all rectangles contribute to the FI integral. Thus the FI can be written as the sum of the FI contribute of each rectangle:

$$\begin{aligned}\mathrm{J}_x &\triangleq \frac{1}{\mathcal{W}_\mathcal{R}}\sum_{n\in\mathcal{N}^o} w_n \mathbb{E}_x\left\{\left(\frac{\partial}{\partial x}\ln\tilde{s}(x, c_n, w_n)\right)^2\right\} \\ &= \frac{1}{\mathcal{W}_\mathcal{R}}\sum_{n\in\mathcal{N}^o} w_n \tilde{\mathrm{J}}_s(w_n) = \frac{1}{\mathcal{W}_\mathcal{R}}\sum_{n\in\mathcal{N}^o}\frac{1}{w_n}\end{aligned} \tag{2}$$

2.3 Derivation of the BFIM for 2-D Maps

Consider a 2-D smoothed uniform map $f(\mathbf{p})$ with support $\mathcal{R} \subset \mathbb{R}^2$; exploiting some definitions given in [1, Sec. II] and assuming that the smoothing along x and y are independent, the pdf for an arbitrary smoothed uniform 2-D map can be put in the form:

$$f(\mathbf{p}) = \frac{1}{\mathcal{A}_\mathcal{R}}\sum_{n\in\mathcal{N}_h^o(y)} s\left(\frac{x-c_{x,n}(y)}{w_n(y)}\right)\cdot\sum_{m\in\mathcal{N}_v^o(x)} s\left(\frac{y-c_{y,m}(x)}{h_m(x)}\right) \tag{3}$$

where $c_{x,n}(y)$ ($c_{y,m}(x)$) and $w_n(y)$ ($h_m(x)$) denote the centre and the length, respectively of the n-th (m-th) segment and where $s(\cdot)$ is a smoothing function as defined in Sec. 2.1. Also note that the regularity condition $\mathbb{E}_{\mathbf{p}}\left\{\frac{\partial \ln f(\mathbf{p})}{\partial \mathbf{p}}\right\} = \mathbf{0}$ is easily verified thanks to the linear operators involved and $s(\cdot)$, which is assumed to respect that condition.

The a priori BFIM, using the iterated expectation and focusing on the FI for the coordinate x, can be written as [2]:

$$[\mathbf{J_p}]_{1,1} \triangleq \mathbb{E}_{\mathbf{p}}\left\{\left(\frac{\partial \ln f(\mathbf{p})}{\partial x}\right)^2\right\} = \mathbb{E}_y\left\{\mathbb{E}_{x|y}\left\{\left(\frac{\partial \ln f(\mathbf{p})}{\partial x}\right)^2\right\}\right\} \tag{4}$$

Then, if we ignore the smoothing for the y coordinate, that is we introduce the approximation

$$f(\mathbf{p}) \approx \frac{1}{\mathcal{A}_{\mathcal{R}}} \sum_{n\in\mathcal{N}_h^o(y)} s\left(\frac{x - c_{x,n}(y)}{w_n(y)}\right) = \frac{\mathcal{W}_{\mathcal{R}}(y)}{\mathcal{A}_{\mathcal{R}}} \frac{1}{\mathcal{W}_{\mathcal{R}}(y)} \sum_{n\in\mathcal{N}_h^o(y)} s\left(\frac{x - c_{x,n}(y)}{w_n(y)}\right)$$

where $\mathcal{W}_{\mathcal{R}}(y) \triangleq \sum_{n\in\mathcal{N}_h^o(y)} w_n(y)$, we reduce the evaluation of the inner expectation to the evaluation of the FI of a 1-D map composed by $N_r = N_h(y)$ segments having widths $\{w_n(y)\}$ and centred around the points $\{c_{x,n}(y)\}$. Thus, using the result obtained in Sec. 2.2, it is easily inferred that

$$\mathbb{E}_{x|y}\left\{\left(\frac{\partial \ln f(\mathbf{p})}{\partial x}\right)^2\right\} \simeq \frac{\mathcal{W}_{\mathcal{R}}(y)}{\mathcal{A}_{\mathcal{R}}} \frac{\mathrm{J}_s}{\mathcal{W}_{\mathcal{R}}(y)} \sum_{n\in\mathcal{N}_h^o(y)} \frac{1}{w_n(y)} \tag{5}$$

Substituting the last result in the RHS of (4) produces

$$[\mathbf{J_p}]_{1,1} \simeq \frac{\mathrm{J}_s}{\mathcal{A}_{\mathcal{R}}} \int_{\mathcal{Y}} \sum_{n\in\mathcal{N}_h^o(y)} \frac{1}{w_n(y)} f(y)dy \simeq \frac{\mathrm{J}_s}{\mathcal{A}_{\mathcal{R}}} \int_{\mathcal{Y}} \sum_{n\in\mathcal{N}_h^o(y)} \frac{dy}{w_n(y)} \tag{6}$$

where $f(y) \triangleq \int f(\mathbf{p})dx$ is the pdf of y only and it has been ignored in the integral since the smoothing is assumed to affect a small portion of the integration domain.

Symmetrically, ignoring the smoothing for the x coordinate, an approximated expression for the FI relative to the coordinate y is obtained. Note however that the two approximations previously mentioned, considered together, are exact only for rectangular maps. Also note that the cross-terms $[\mathbf{J_p}]_{2,1}$ and $[\mathbf{J_p}]_{1,2}$ of the BFIM are non-zero if and only if the parameters x and y are independent (like in a 2-D rectangle); in general, because of the smoothing this is not exactly true, but such dependence is typically weak, so that a good approximation for the BFIM is given by:

$$\mathbf{J_p} \simeq \frac{\mathrm{J}_s}{\mathcal{A}_{\mathcal{R}}} \operatorname{diag}\left\{\int_{\mathcal{Y}} \sum_{n\in\mathcal{N}_h^o(y)} \frac{dy}{w_n(y)}, \int_{\mathcal{X}} \sum_{m\in\mathcal{N}_v^o(x)} \frac{dx}{h_m(x)}\right\} \tag{7}$$

which coincides with [1, Eq. (8)].

Note that the BCRB for a generic estimator $\hat{\mathbf{p}}(\mathbf{z})$ of $\mathbf{p}$ based on the observation vector $\mathbf{z}$ is given by

$$\boldsymbol{\Lambda}\left(\hat{\mathbf{p}}(\mathbf{z})\right) \triangleq \mathbb{E}_{\hat{\mathbf{p}}(\mathbf{z}),\mathbf{p}}\left\{\left(\hat{\mathbf{p}}(\mathbf{z})-\mathbf{p}\right)\left(\hat{\mathbf{p}}(\mathbf{z})-\mathbf{p}\right)^T\right\} \succeq \mathbf{J}^{-1}$$

and is obtained from (7) assuming a specific observation model which connects $\mathbf{z}$ with $\mathbf{p}$ and employing the relation:

$$\mathbf{J} = \mathbf{J}_{\mathbf{z}|\mathbf{p}} + \mathbf{J}_{\mathbf{p}}$$

where $\mathbf{J}_{\mathbf{z}|\mathbf{p}} \triangleq \mathbb{E}_{\mathbf{z},\mathbf{p}}\left\{-\frac{\partial}{\partial \mathbf{p}}\left[\frac{\partial}{\partial \mathbf{p}} \ln f(\mathbf{z}|\mathbf{p})\right]^T\right\}$ and $\mathbf{J}_{\mathbf{p}}$ is given, in this contest, by (7).

3 Derivation of the EZZB for Generic-Shape Uniform Maps

In this Section we prove the EZZB expression presented in [1, Eq. (12-13)] using the derivation of [3, Sec. II.B] as guideline. Then the EZZB derived is evaluated for 1-D maps and extended to the 2-D case, resulting in [1, Eq. (14)].

3.1 Derivation of the EZZB

Consider a random vector $\mathbf{p} \in \mathcal{R} \subset \mathbb{R}^2$; the Bayesian mean square error matrix (BMSE) associated with a generic estimator $\hat{\mathbf{p}}(\mathbf{z})$ of $\mathbf{p}$ based on the observation vector $\mathbf{z}$ is given by

$$\boldsymbol{\Lambda}\left(\hat{\mathbf{p}}(\mathbf{z})\right) \triangleq \mathbb{E}_{\hat{\mathbf{p}}(\mathbf{z}),\mathbf{p}}\left\{\left(\hat{\mathbf{p}}(\mathbf{z})-\mathbf{p}\right)\left(\hat{\mathbf{p}}(\mathbf{z})-\mathbf{p}\right)^T\right\}$$

If the identity [4, p. 24] is considered, then:

$$E_\nu \triangleq \left[\boldsymbol{\Lambda}\left(\hat{\mathbf{p}}(\mathbf{z})\right)\right]_\nu = \frac{1}{2}\int_0^\infty \Pr\left\{|\xi_\nu| \geq \frac{h}{2}\right\} h\, dh \tag{8}$$

where E_ν is the estimation error for the coordinate ν, $\xi_\nu = \left[\hat{\mathbf{p}}(\mathbf{z})-\mathbf{p}\right]_\nu$, $\hat{\mathbf{p}}(\mathbf{z})$ is some generic estimator of the r.v. $\mathbf{p}$ based on the observation vector $\mathbf{z}$ and $\nu \in \{x, y\}$. Then, trivially $\Pr\left\{|\xi_\nu| \geq \frac{h}{2}\right\} = \Pr\left\{\xi_\nu > \frac{h}{2}\right\} + \Pr\left\{\xi_\nu \leq -\frac{h}{2}\right\}$; if we consider two events with non-zero probability $\mathbf{p} = \boldsymbol{\rho}_0$ and $\mathbf{p} = \boldsymbol{\rho}_1$, we can express the inner term of (8) as:

$$\Pr\left\{|\xi_\nu| \geq \frac{h}{2}\right\} = \int_{\mathbb{R}^2} \Pr\left\{\xi_\nu > \frac{h}{2}|\mathbf{p} = \boldsymbol{\rho}_0\right\} f(\boldsymbol{\rho}_0) d\boldsymbol{\rho}_0 + \int_{\mathbb{R}^2} \Pr\left\{\xi_\nu \leq -\frac{h}{2}|\mathbf{p} = \boldsymbol{\rho}_1\right\} f(\boldsymbol{\rho}_1) d\boldsymbol{\rho}_1 \tag{9}$$

where $f(\cdot)$ is the a-priori pdf for the parameter to estimate $\mathbf{p}$. If we let $\boldsymbol{\rho}_0 = \boldsymbol{\rho}$ and $\boldsymbol{\rho}_1 = \boldsymbol{\rho} + \boldsymbol{\delta}(h)$, where $\boldsymbol{\delta}(h)$ is some function of the integration variable of (8) and we constrain the integrals of (9) on a subset $\mathcal{R}(h)$ of the map support $\mathcal{R}$, such that $f(\boldsymbol{\rho}) > 0$ and $f(\boldsymbol{\rho} + \boldsymbol{\delta}(h)) > 0$ (and thus the conditional probabilities of (9) are meaningful), then we can divide and multiply by the quantity $f(\boldsymbol{\rho}_0) + f(\boldsymbol{\rho}_1) = f(\boldsymbol{\rho}) + f(\boldsymbol{\rho} + \boldsymbol{\delta}(h)) > 0$ and obtain the same result reported in [3, Eq. (22-30)]:

$$\Pr\left\{|\xi_\nu| \geq \frac{h}{2}\right\} \geq \int_{\mathcal{R}(h)} \left[f(\boldsymbol{\rho}) + f(\boldsymbol{\rho} + \boldsymbol{\delta}(h))\right] P^{\mathbf{z}}_{\min}(\boldsymbol{\rho}, \boldsymbol{\rho} + \boldsymbol{\delta}(h)) d\boldsymbol{\rho} \tag{10}$$

which holds for any $\boldsymbol{\delta}(h)$ satisfying the equality $\mathbf{a}^T\boldsymbol{\delta}(h) = h$, where $\mathbf{a} \in \mathbb{R}^2$. Note that (10) generalizes [3, Eq. (30)] to the case where the pdf $f(\cdot)$ of the parameter to estimate has a bounded support. Here $\mathbf{a} = \mathbf{e}_\nu$ is selected so that $\boldsymbol{\delta}(h) = h\mathbf{e}_\nu$; this choice allows us to obtain a bound on the estimation error variances. Then substituting (10) in (8) produces [1, Eq. (13)], which is reported here for reader convenience:

$$E_\nu \geq Z_\nu \triangleq \frac{1}{2} \iint_{\mathcal{P}_\nu} \left[f\left(\boldsymbol{\rho}\right) + f\left(\boldsymbol{\rho} + h\mathbf{e}_\nu\right)\right] P^{\mathbf{z}}_{\min}\left(\boldsymbol{\rho}, \boldsymbol{\rho} + h\mathbf{e}_\nu\right) h \, d\boldsymbol{\rho} dh \tag{11}$$

where $\nu \in \{x, y\}$; note that the integration domains of h and $\boldsymbol{\rho}$ have been merged in the set $\mathcal{P}_\nu$:

$$\mathcal{P}_\nu \triangleq \{(h, \boldsymbol{\rho}) : h \geq 0 \wedge f\left(\boldsymbol{\rho}\right) > 0 \wedge f\left(\boldsymbol{\rho} + h\mathbf{e}_\nu\right) > 0\} \subset \mathbb{R}^3$$

and that the complete bound is written as

$$\boldsymbol{\Lambda}\left(\hat{\mathbf{p}}(\mathbf{z})\right) \succeq \mathbf{Z} \triangleq \mathrm{diag}\left\{Z_x, Z_y\right\}$$

or equivalently

$$\mathbf{E} \triangleq \mathrm{diag}\left\{E_x, E_y\right\} \succeq \mathbf{Z} \tag{12}$$

3.2 Derivation of the EZZB for 1-D Maps

Let us evaluate now (11) for a 1-D uniform map whose support $\mathcal{R} \subset \mathbb{R}$ consists of the union of N_r disjoint segments and where the agent position to estimate is the scalar x. In the following such segments are indexed by the odd numbers of the set $\mathcal{N}^o = \{1, 3, ..., 2N_r - 1\}$ and the lower (upper) limit of the segment $n \in \mathcal{N}^o$ is denoted $l_n \triangleq c_n - \frac{w_n}{2}$ ($u_n \triangleq c_n + \frac{w_n}{2}$), where c_n (w_n) represents the centre (length) of the segment itself. Moreover, it is assumed, without any loss of generality, that $c_1 < c_3 < ... < c_{2N_r-1}$.

Note that in this context, the smoothing function which is required for the BFIM analysis (see Appendix2.1) is not used, and the truly uniform map model is adopted. Eq. (11) can be adapted to 1-D scenario:

$$Z \triangleq \frac{1}{2} \iint_{\mathcal{P}} \left[f(\tau) + f(\tau + h)\right] P^{z}_{\min}(\tau, \tau + h) h \, d\tau dh \tag{13}$$

where $\boldsymbol{\rho} = \tau$, $\mathcal{P} = \{(\tau, h) : h \geq 0 \wedge f(\tau) > 0 \wedge f(\tau + h) > 0\}$ and $P^z_{\min}(\tau, \tau + h)$ represents the minimum error probability of a binary detector which, on the basis of a noisy datum z and a likelihood ratio test, has to select one of the following two hypotheses $H_0 : x = \tau$ and $H_1 : x = \tau + h$.

We note that: a) because of our uniform map assumption, for any $(\tau, h) \in \mathcal{P}$, $f(\tau) + f(\tau + h) = \frac{2}{\mathcal{W}_{\mathcal{R}}}$ (with $\mathcal{W}_{\mathcal{R}} \triangleq \sum_{n \in \mathcal{N}^o} w_n$), because of the uniformity of the considered map; b) the prior-probabilities of the hypotheses H_0 and H_1 are the same and the maximum-likelihood rule is thus the optimal detection rule: $\hat{H}_{opt}(z) = \arg\max_{H \in \{H_0, H_1\}} f(z|H)$. Assuming a Gaussian observation model $z = x + n$, where $n \sim \mathcal{N}(0, \sigma^2)$, we obtain $f(z|H = H_0) = \mathcal{N}(z; \tau, \sigma^2)$ and $f(z|H = H_1) = \mathcal{N}(z; \tau + h, \sigma^2)$. The optimal detection rule thus becomes

$$\hat{H}_{opt}(z) = \begin{cases} H_0 & z \leq t + \frac{h}{2} \\ H_1 & z > t + \frac{h}{2} \end{cases}$$

and its probability of error is $P^z_{\min}(\tau, \tau + h) = \frac{1}{2} \operatorname{erfc}\left(\frac{h}{2\sigma\sqrt{2}}\right)$, so that $P^z_{\min}(\tau, \tau + h)$ is not a function of τ and (13) further simplifies to:

$$\begin{aligned} Z &= \frac{1}{2\mathcal{W}_{\mathcal{R}}} \iint_{\mathcal{P}} h \operatorname{erfc}\left(\frac{h}{2\sigma\sqrt{2}}\right) d\tau\, dh \\ &= \frac{\sigma^3}{2\mathcal{W}_{\mathcal{R}}} \iint_{\mathcal{Q}} u \operatorname{erfc}\left(\frac{u}{2\sqrt{2}}\right) dt\, du \end{aligned} \tag{14}$$

where $u \triangleq h/\sigma$, $t \triangleq \tau/\sigma$ and $\mathcal{Q} \triangleq \{(t, u) : u \geq 0 \wedge f(\sigma t) > 0 \wedge f(\sigma(t + u)) > 0\}$ can be shown to be a 2-D domain consisting of the union of N_r triangles and $N_r(N_r - 1)/2$ parallelograms in the plane (t, u), as exemplified by Fig. 1, which refers to the case $N_r = 3$. In particular, the contribution to (14) from the i-th triangle $\mathcal{Q}_i \triangleq \left\{(t, u) : \frac{l_i}{\sigma} \leq t \leq \frac{u_i}{\sigma} \wedge 0 \leq u \leq \frac{u_i}{\sigma} - t\right\}$ is

$$\begin{aligned} \frac{\sigma^3}{2\mathcal{W}_{\mathcal{R}}} \iint_{\mathcal{Q}_i} u \operatorname{erfc}\left(\frac{u}{2\sqrt{2}}\right) dt\, du &= \frac{\sigma^3}{2\mathcal{W}_{\mathcal{R}}} \int_0^{\frac{u_i}{\sigma} - t} \int_{\frac{l_i}{\sigma}}^{\frac{u_i}{\sigma}} u \operatorname{erfc}\left(\frac{u}{2\sqrt{2}}\right) dt\, du \\ &= \frac{\sigma^3}{2\mathcal{W}_{\mathcal{R}}} \int_0^{\rho_i} \int_{\frac{l_i}{\sigma}}^{\frac{u_i}{\sigma} - u} u \operatorname{erfc}\left(\frac{u}{2\sqrt{2}}\right) dt\, du \\ &= \frac{\sigma^3}{2\mathcal{W}_{\mathcal{R}}} \int_0^{\rho_i} (\rho_i - u)\, u \operatorname{erfc}\left(\frac{u}{2\sqrt{2}}\right) dt\, du \end{aligned}$$

which can be written in a compact form as $\frac{\sigma^2}{2\rho_{\mathcal{W}}} \zeta(\rho_i)$, where $\rho_i \triangleq w_i/\sigma$, $\rho_{\mathcal{W}} \triangleq \mathcal{W}_{\mathcal{R}}/\sigma$, the function $\zeta(\rho)$ is defined as

$$\zeta(\rho) \triangleq \int_0^{\rho} (\rho - u) u \operatorname{erfc}\left(\frac{u}{2\sqrt{2}}\right) du \tag{15}$$

and $i \in \mathcal{N}^o$.

As far as the parallelograms are concerned, the i-th map support segment, for some values of h, will overlap with all previous segments[1], generating the parallelograms shown in Fig. 1; let the overlap of the i-th segment with the $(i-1)$-th segment define the subset $\mathcal{Q}_{ov,i} \triangleq \left\{(t,u) : \frac{l_i}{\sigma} \leq t \leq \frac{u_i}{\sigma} \wedge \frac{l_i}{\sigma} - t \leq u \leq \frac{u_i}{\sigma} - t\right\}$, with $i \in \mathcal{N}^e \triangleq \{2, 4, ..., 2(N_r - 1)\}$; then the integral of (14) over $\mathcal{Q}_{ov,i}$ contributes the term $\frac{\sigma^2}{2\rho_{\mathcal{W}}} \zeta_{ov}(\rho_{\triangle_i}, \rho_{i-1}, \rho_i)$, where $\rho_{\triangle_i} \triangleq \triangle x_i / \sigma_w$, $\triangle x_i \triangleq l_i - u_{i-1}$ and $\zeta_{ov}(\rho_\triangle, \rho_1, \rho_2)$ is defined as

$$\begin{aligned}\zeta_{ov}\left(\rho_\triangle, \rho_1, \rho_2\right) \triangleq & \int_{\rho_\triangle}^{\rho_\triangle + \rho_2} (u - \rho_\triangle) u \operatorname{erfc}\left(\frac{u}{2\sqrt{2}}\right) du + \\ & \int_{\rho_\triangle + \rho_2}^{\rho_\triangle + \rho_1} \rho_2 u \operatorname{erfc}\left(\frac{u}{2\sqrt{2}}\right) du + \\ & \int_{\rho_\triangle + \rho_1}^{\rho_\triangle + \rho_1 + \rho_2} (\rho_\triangle + \rho_1 + \rho_2 - u) u \operatorname{erfc}\left(\frac{u}{2\sqrt{2}}\right) du \end{aligned} \tag{16}$$

for $\rho_1 > \rho_2$ (it can be shown that $\zeta_{ov}(\rho_\triangle, \rho_1, \rho_2) = \zeta_{ov}(\rho_\triangle, \rho_2, \rho_1)$).

We then note that the contributes to (14) of the overlaps of the i-th segment with the $(i-2)$-th, $(i-3)$-th, ..., 1st segment are always positive since the integrand $u \operatorname{erfc}\left(\frac{u}{2\sqrt{2}}\right) \geq 0$, given that $u \geq 0$. Thus neglecting such regions of the plane (u, t), i.e. restricting $\mathcal{Q}$ to $\left(\bigcup_{n=1}^{N_r} \mathcal{Q}_n\right) \bigcup \left(\bigcup_{n=1}^{N_r - 1} \mathcal{Q}_{ov,n}\right)$, a lower bound on the EZZB Z is obtained. This choice allows us to derive a bound with a tractable analytical expression: summing together the N_r contributes of (15) and the $N_r - 1$ contributes of (16), the lower bound becomes:

$$E \triangleq \lambda\left(\hat{x}(z)\right) \geq Z \geq \frac{\sigma^2}{2\rho_{\mathcal{W}}} \left[\sum_{n \in \mathcal{N}^o} \zeta\left(\rho_n\right) + \sum_{n \in \mathcal{N}^e} \zeta_{ov}\left(\rho_{\triangle_n}, \rho_{n-1}, \rho_{n+1}\right)\right] \tag{17}$$

where $\lambda\left(\hat{x}(z)\right) \triangleq \mathbb{E}_{\hat{x}(z),x}\left\{\left(\hat{x}(z) - x\right)\left(\hat{x}(z) - x\right)^T\right\}$ is the BMSE for the 1-D estimator $\hat{x}(z)$.

[1]It is more formally accurate to write that, decomposing $f(\tau)$ as $f_1(\tau) + f_2(\tau)$, where $f_1(\tau)$ is the pdf associated with the first segment parametrized by (c_1, w_1) and $f_2(\tau)$ is the pdf similarly associated with the second segment (c_2, w_2), then the support of $f_2(\tau + h)$ will overlap with the support of $f_1(\tau)$ for some values of h. In the following however, for the sake of brevity, we will refer to the "overlap of the pdf supports" just as the "overlap of the segments".

3.3 Derivation of the EZZB for 2-D Maps

Consider a 2-D uniform map $f(\mathbf{p}) = f(x, y)$ with support $\mathcal{R} \subset \mathbb{R}^2$; writing (11) for $\nu = x$, the EZZB is written as

$$E_x \geq Z_x = \frac{1}{2} \iiint_{\mathcal{P}_x} [f(\tau, \upsilon) + f(\tau + h, \upsilon)] \, P^{\mathbf{z}}_{\min}(\boldsymbol{\rho}, \boldsymbol{\rho} + h\mathbf{e}_x) h \, d\tau \, d\upsilon \, dh \tag{18}$$

where $\mathcal{P}_x \triangleq \{(h, \tau, \upsilon) : h \geq 0 \wedge f(\tau, \upsilon) > 0 \wedge f(\tau + h, \upsilon) > 0\} \subset \mathbb{R}^3$ and $\boldsymbol{\rho} \triangleq [\tau, \upsilon]^T$. Note that $\mathcal{P}_x$ is the straightforward extension of the integration domain $\mathcal{P}$ appearing in Sec. 3.2. The minimum error probability $P^{\mathbf{z}}_{\min}(\boldsymbol{\rho}, \boldsymbol{\rho} + h\mathbf{e}_x)$, adopting the Gaussian observation model $f(\mathbf{z}|\mathbf{p}) = \mathcal{N}(\mathbf{z}; \mathbf{p}, \boldsymbol{\Sigma})$, with $\boldsymbol{\Sigma} = \text{diag}\left\{\sigma_x^2, \sigma_y^2\right\}$, is computed using the optimal detection rule $\hat{H}_{opt,x}(\mathbf{z})$, whose expression is similar to the one obtained for 1-D maps:

$$\hat{H}_{opt,x}(\mathbf{z}) = \begin{cases} H_0 & z_x \leq x + \frac{h}{2} \\ H_1 & z_x > x + \frac{h}{2} \end{cases}$$

where $\mathbf{z} \triangleq [z_x, z_y]^T$. Its probability of error is thus, unsurprisingly, the same as in the 1-D case: $P^{\mathbf{z}}_{\min}(\boldsymbol{\rho}, \boldsymbol{\rho} + h\mathbf{e}_x) = \frac{1}{2} \text{erfc}\left(\frac{h}{2\sigma_x\sqrt{2}}\right) = P^{z}_{\min}(\tau, \tau + h)$. Furthermore, with few regularity assumptions on the map support, the integral over $\mathcal{P}_x$ can be decomposed as the integral over (h, x) and y:

$$Z_x = \int_{\mathcal{Y}} \left\{ \frac{1}{2} \iint_{\mathcal{P}_x(\upsilon)} [f(\tau, \upsilon) + f(\tau + h, \upsilon)] \, P^{z}_{\min}(\tau, \tau + h) h \, d\tau \, dh \right\} d\upsilon \tag{19}$$

where $\mathcal{P}_x(\upsilon) = \{(h, \tau) : h \geq 0 \wedge f(\tau, \upsilon) > 0 \wedge f(\tau + h, \upsilon) > 0\}$ is the slice of $\mathcal{P}_x \subset \mathbb{R}^3$ at coordinate υ and the expression inside the curly brackets is recognized to be the EZZB for a 1-D map (see (13)), function of the integration variable υ, whose number of segments $N_r(\upsilon) = N_h(\upsilon)$ and the lengths of the 1-D segments are given by $\{w_n(\upsilon)\}$; the distance between the $(n+1)$-th and the n-th segment at ordinate υ is then $\triangle w_n(\upsilon)$. Thus, plugging (17) into (19) it is easy to obtain:

$$Z_x = \int_{\mathcal{Y}} \left\{ \frac{\sigma_x^3}{2\mathcal{A}_{\mathcal{R}}} \left[\sum_{n \in \mathcal{N}_h^o(\upsilon)} \zeta\left(\frac{w_n(\upsilon)}{\sigma_x}\right) + \sum_{n \in \mathcal{N}_h^e(\upsilon)} \zeta_{ov}\left(\frac{\triangle w_n(\upsilon)}{\sigma_x}, \frac{w_{n-1}(\upsilon)}{\sigma_x}, \frac{w_{n+1}(\upsilon)}{\sigma_x}\right) \right] \right\} d\upsilon$$

Repeating the proof for the dual coordinate $\nu = y$, the EZZB expression

$$\mathbf{E} \succeq \mathbf{Z} = \text{diag}\Bigg\{$$

$$\frac{\sigma_x^3}{2\mathcal{A}_{\mathcal{R}}} \int_{\mathcal{Y}} \left[\sum_{n \in \mathcal{N}_h^o(y)} \zeta\left(\frac{w_n(y)}{\sigma_x}\right) + \sum_{n \in \mathcal{N}_h^e(y)} \zeta_{ov}\left(\frac{\triangle w_n(y)}{\sigma_x}, \frac{w_{n-1}(y)}{\sigma_x}, \frac{w_{n+1}(y)}{\sigma_x}\right) \right] dy,$$

$$\frac{\sigma_y^3}{2\mathcal{A}_{\mathcal{R}}}\int_{\mathcal{X}}\left[\sum_{m\in\mathcal{N}_v^o(x)}\zeta\left(\frac{h_m(x)}{\sigma_y}\right)+\sum_{m\in\mathcal{N}_v^e(x)}\zeta_{ov}\left(\frac{\triangle h_m(x)}{\sigma_y},\frac{h_{m-1}(x)}{\sigma_y},\frac{h_{m+1}(x)}{\sigma_y}\right)\right]dx\right\} \quad (20)$$

is found, where $\mathbf{E} \triangleq \text{diag}\{E_x, E_y\}$. Note that (20) coincides with [1, Eq. (14)].

4 Derivation of the WWB for Generic-Shape Uniform Maps

In this Section we prove the WWB expression presented in [1, Eq. (20)] using the derivation of [5] as guideline. Then the WWB is derived for N-D maps and finally specialized for 2-D case, resulting in [1, Eq. (22-26)].

4.1 Derivation of the WWB

Consider a random vector $\mathbf{p} \in \mathcal{R} \subset \mathbb{R}^N$; the Bayesian mean square error matrix (BMSE) $\mathbf{\Lambda}(\hat{\mathbf{p}}(\mathbf{z}))$ associated with a generic estimator $\hat{\mathbf{p}}(\mathbf{z})$ of $\mathbf{p}$ based on the observation vector $\mathbf{z}$ is given by

$$\mathbf{\Lambda}(\hat{\mathbf{p}}(\mathbf{z})) \triangleq \mathbb{E}_{\hat{\mathbf{p}}(\mathbf{z}),\mathbf{p}}\left\{(\hat{\mathbf{p}}(\mathbf{z})-\mathbf{p})(\hat{\mathbf{p}}(\mathbf{z})-\mathbf{p})^T\right\}$$

The WWB on such estimation error covariance is given by [5, Eq. (7)]:

$$\mathbf{\Lambda}(\hat{\mathbf{p}}(\mathbf{z})) \succeq \mathbf{H}\mathbf{G}^{-1}\mathbf{H}^T \quad (21)$$

where $\mathbf{H} = [\mathbf{h}_1, ..., \mathbf{h}_N]$ is a matrix of test vectors $\mathbf{h}_i \in \mathbb{R}^N$ and the matrix $\mathbf{G}$ is given by [5, Eq. (8)]

$$[\mathbf{G}]_{i,j} = \frac{\mathbb{E}_{\mathbf{z},\mathbf{p}}\{r(\mathbf{z},\mathbf{p};\mathbf{h}_i,s_i)\, r(\mathbf{z},\mathbf{p};\mathbf{h}_j,s_j)\}}{\mathbb{E}_{\mathbf{z},\mathbf{p}}\{L^{s_i}(\mathbf{z};\mathbf{p}+\mathbf{h}_i,\mathbf{p})\}\,\mathbb{E}_{\mathbf{z},\mathbf{p}}\{L^{s_j}(\mathbf{z};\mathbf{p}+\mathbf{h}_j,\mathbf{p})\}} \quad (22)$$

where

$$r(\mathbf{z},\mathbf{p};\mathbf{h}_i,s_i) \triangleq L^{s_i}(\mathbf{z};\mathbf{p}+\mathbf{h}_i,\mathbf{p}) - L^{1-s_i}(\mathbf{z};\mathbf{p}-\mathbf{h}_i,\mathbf{p}) \quad (23)$$

and s_i, with $i = 1, ..., N$ is a scalar optimization parameter. The inequality (21) holds for any couple $\{\mathbf{h}_i, s_i\}$with $i = 1, ..., N$, such that $\mathbf{G}$ is well defined and invertible.

The tightest bound is obtained optimizing against the $\{\mathbf{h}_i, s_i\}$ parameters:

$$\mathbf{\Lambda}(\hat{\mathbf{p}}(\mathbf{z})) \succeq \sup_{\{\mathbf{h}_i,s_i\}} \mathbf{H}\mathbf{G}^{-1}\mathbf{H}^T \quad (24)$$

In order to simplify the WWB analytical evaluation, we set $s_i = \frac{1}{2}$, $\forall i$ (as suggested by Weiss and Weinstein this choice typically provides a tight bound) and $\mathbf{H} = \text{diag}\{h_1, ..., h_N\}$ to obtain a bound only on the error variances (and not covariances). It is easy to show that with these choices, the elements off-diagonal of $\mathbf{H}\mathbf{G}^{-1}\mathbf{H}^T$ are zero while the i-th term of the diagonal is (see (22)):

$$\left[\mathbf{H}\mathbf{G}^{-1}\mathbf{H}^T\right]_{i,i} = h_i^2 \frac{\left(\mathbb{E}_{\mathbf{z},\mathbf{p}}\left\{L^{\frac{1}{2}}\left(\mathbf{z};\mathbf{p}+\mathbf{h}_i,\mathbf{p}\right)\right\}\right)^2}{\mathbb{E}_{\mathbf{z},\mathbf{p}}\left\{r^2\left(\mathbf{z},\mathbf{p};\mathbf{h}_i,s_i\right)\right\}} \tag{25}$$

The WWB (24) thus may be re-written as

$$\mathbf{\Lambda}\left(\hat{\mathbf{p}}(\mathbf{z})\right) \succeq \mathbf{W} \tag{26}$$

where $\mathbf{W} = \text{diag}\left\{W_1, ..., W_N\right\}$ and the generic diagonal term W_ν is (see (23) and (25)):

$$W_\nu \triangleq \sup_{h_\nu \in \mathbb{R}} \frac{\left(h_\nu \mathbb{E}_{\mathbf{z},\mathbf{p}}\left\{L^{\frac{1}{2}}\left(\mathbf{z};\mathbf{p}+h_\nu\mathbf{e}_\nu,\mathbf{p}\right)\right\}\right)^2}{\mathbb{E}_{\mathbf{z},\mathbf{p}}\left\{\left[L^{\frac{1}{2}}\left(\mathbf{z};\mathbf{p}+h_\nu\mathbf{e}_\nu,\mathbf{p}\right) - L^{\frac{1}{2}}\left(\mathbf{z};\mathbf{p}-h_\nu\mathbf{e}_\nu,\mathbf{p}\right)\right]^2\right\}} \tag{27}$$

which coincides with [1, Eq. (20)]. If $\mathbf{E} \triangleq \text{diag}\left\{E_1, ..., E_N\right\}$ is defined, with $E_\nu = \left[\mathbf{\Lambda}\left(\hat{\mathbf{p}}(\mathbf{z})\right)\right]_\nu$, then (26) is obviously equivalent to

$$\mathbf{E} \succeq \mathbf{W}$$

4.2 Derivation of the WWB for N-D Maps

Consider a N-D uniform map $f(\mathbf{p})$ with support $\mathcal{R} \subset \mathbb{R}^N$; the WWB term associated to the coordinate ν is (27), where $\nu \in \{x, y, z, ...\}$. Adopting the Gaussian observation model

$$f(\mathbf{z}|\mathbf{p}) = \mathcal{N}\left(\mathbf{z};\mathbf{p},\mathbf{\Sigma}\right) \tag{28}$$

with $\mathbf{\Sigma} = \text{diag}\left\{\sigma_1^2, ..., \sigma_N^2\right\}$, the likelihood ratios appearing in (27) become $L\left(\mathbf{z};\mathbf{p}_1,\mathbf{p}_2\right) = \frac{\mathcal{N}(\mathbf{z};\mathbf{p}_1,\mathbf{\Sigma})}{\mathcal{N}(\mathbf{z};\mathbf{p}_2,\mathbf{\Sigma})}$ $\forall \mathbf{p}_1, \mathbf{p}_2 \in \mathcal{R}$. Thus the numerator of (27) may be written as

$$\begin{aligned}\mathbb{E}_{\mathbf{z},\mathbf{p}}\left\{L^{\frac{1}{2}}\left(\mathbf{z};\mathbf{p}+h_\nu\mathbf{e}_\nu,\mathbf{p}\right)\right\} &= \frac{1}{\mathcal{A}_\mathcal{R}} \int_{\mathbb{R}^N} \int_{\mathcal{P}_\nu(h_\nu)} \mathcal{N}^{\frac{1}{2}}\left(\mathbf{z};\boldsymbol{\rho}+h_\nu\mathbf{e}_\nu,\mathbf{\Sigma}\right) \mathcal{N}^{\frac{1}{2}}\left(\mathbf{z};\boldsymbol{\rho},\mathbf{\Sigma}\right) d\boldsymbol{\rho}\, d\mathbf{z} \\ &= \frac{1}{\mathcal{A}_\mathcal{R}} \exp\left(-\frac{h_\nu^2}{8\sigma_\nu^2}\right) \int_{\mathcal{P}_\nu(h_\nu)} d\boldsymbol{\rho} \\ &= \frac{1}{\mathcal{A}_\mathcal{R}} \exp\left(-\frac{h_\nu^2}{8\sigma_\nu^2}\right) \lambda_\nu\left(h_\nu,\mathcal{R}\right) \end{aligned} \tag{29}$$

where $\mathcal{P}_\nu(h_\nu) \triangleq \{\boldsymbol{\rho} : f(\boldsymbol{\rho}) > 0 \wedge f(\boldsymbol{\rho} + h_\nu \mathbf{e}_\nu) > 0\}$ is a slice of the integration domain $\mathcal{P}_\nu \subset \mathbb{R}^{N+1}$ (which is the same domain involved in the evaluation of the EZZB when $N = 2$), at coordinate h in the $(h; \boldsymbol{\rho})$ space and the function $\lambda_\nu(h_\nu, \mathcal{R})$ is defined as

$$\lambda_\nu(h_\nu, \mathcal{R}) \triangleq \int \mathbb{I}_\mathcal{R}(\mathbf{p}) \mathbb{I}_\mathcal{R}(\mathbf{p} + h_\nu \mathbf{e}_\nu) d\mathbf{p} = \int_{\mathcal{P}_\nu(h_\nu)} d\boldsymbol{\rho} \tag{30}$$

The expectation appearing in the denominator of (27) can be expressed as

$$\begin{aligned}
\mathbb{E}_{\mathbf{z},\mathbf{p}} &\left\{ \left[L^{\frac{1}{2}}(\mathbf{z}; \mathbf{p} + h_\nu \mathbf{e}_\nu, \mathbf{p}) - L^{\frac{1}{2}}(\mathbf{z}; \mathbf{p} - h_\nu \mathbf{e}_\nu, \mathbf{p}) \right]^2 \right\} = \\
&\mathbb{E}_{\mathbf{z},\mathbf{p}} \{ L(\mathbf{z}; \mathbf{p} + h_\nu \mathbf{e}_\nu, \mathbf{p}) \} + \mathbb{E}_{\mathbf{z},\mathbf{p}} \{ L(\mathbf{z}; \mathbf{p} - h_\nu \mathbf{e}_\nu, \mathbf{p}) \} - \\
&\qquad 2\mathbb{E}_{\mathbf{z},\mathbf{p}} \left\{ L^{\frac{1}{2}}(\mathbf{z}; \mathbf{p} + h_\nu \mathbf{e}_\nu, \mathbf{p}) L^{\frac{1}{2}}(\mathbf{z}; \mathbf{p} - h_\nu \mathbf{e}_\nu, \mathbf{p}) \right\}
\end{aligned}$$

i.e., as the sum of three terms which are denoted A, B and C, respectively, in the following. It is important to note that

$$\begin{aligned}
A &= \mathbb{E}_{\mathbf{z},\mathbf{p}} \{ L(\mathbf{z}; \mathbf{p} + h_\nu \mathbf{e}_\nu, \mathbf{p}) \} \\
&= \frac{1}{\mathcal{A}_\mathcal{R}} \int_{\mathbb{R}^2} \int_{\mathcal{P}_\nu(h_\nu)} \mathcal{N}(\mathbf{z}; \boldsymbol{\rho} + h_\nu \mathbf{e}_\nu, \boldsymbol{\Sigma}) d\boldsymbol{\rho}\, d\mathbf{z} \\
&= \frac{1}{\mathcal{A}_\mathcal{R}} \int_{\mathcal{P}_\nu(h_\nu)} d\boldsymbol{\rho} = \frac{1}{\mathcal{A}_\mathcal{R}} \lambda_\nu(h_\nu, \mathcal{R}) \\
&= \mathbb{E}_{\mathbf{z},\mathbf{p}} \{ L(\mathbf{z}; \mathbf{p} - h_\nu \mathbf{e}_\nu, \mathbf{p}) \} = B
\end{aligned} \tag{31}$$

and

$$\begin{aligned}
C &= \frac{1}{\mathcal{A}_\mathcal{R}} \int_{\mathbb{R}^2} \int_{\tilde{\mathcal{P}}_\nu(h_\nu)} \mathcal{N}(\mathbf{z}; \boldsymbol{\rho} + h_\nu \mathbf{e}_\nu, \boldsymbol{\Sigma}) d\boldsymbol{\rho}\, d\mathbf{z} \\
&= \frac{1}{\mathcal{A}_\mathcal{R}} \exp\left(-\frac{h_\nu^2}{2\sigma_\nu^2} \right) \int_{\tilde{\mathcal{P}}_\nu(h_\nu)} d\boldsymbol{\rho} \\
&= \frac{1}{\mathcal{A}_\mathcal{R}} \exp\left(-\frac{h_\nu^2}{2\sigma_\nu^2} \right) \gamma_\nu(h_\nu, \mathcal{R})
\end{aligned} \tag{32}$$

where $\tilde{\mathcal{P}}_\nu(h_\nu) \triangleq \{\boldsymbol{\rho} : f(\boldsymbol{\rho}) > 0 \wedge f(\boldsymbol{\rho} + h\mathbf{e}_\nu) > 0 \wedge f(\boldsymbol{\rho} - h\mathbf{e}_\nu) > 0\}$, and the function $\gamma_\nu(h_\nu, \mathcal{R})$ is defined as

$$\gamma_\nu(h_\nu, \mathcal{R}) \triangleq \int \mathbb{I}_\mathcal{R}(\mathbf{p}) \mathbb{I}_\mathcal{R}(\mathbf{p} + h_\nu \mathbf{e}_\nu) \mathbb{I}_\mathcal{R}(\mathbf{p} - h_\nu \mathbf{e}_\nu) d\mathbf{p} = \int_{\tilde{\mathcal{P}}_\nu(h_\nu)} d\boldsymbol{\rho} \tag{33}$$

Finally, substituting (29), (31) and (32) in (27), we obtain that the WWB for the Gaussian observation model (28) and the N-D map considered, has the equivalent form

$$W_\nu = \sup_{h_\nu \in \mathbb{R}} \frac{\left[\frac{1}{\mathcal{A}_\mathcal{R}} \exp\left(-\frac{h_\nu^2}{8\sigma_\nu^2}\right) \lambda_\nu\left(h_\nu, \mathcal{R}\right)\right]^2}{\frac{2}{\mathcal{A}_\mathcal{R}} \lambda_\nu\left(h_\nu, \mathcal{R}\right) + \frac{1}{\mathcal{A}_\mathcal{R}} \exp\left(-\frac{h_\nu^2}{2\sigma_\nu^2}\right) \gamma_\nu\left(h_\nu, \mathcal{R}\right)} \tag{34}$$

Note that (34) coincides with [1, Eq. (22)].

4.3 Derivation of the WWB for 2-D Maps

The functions $\lambda_\nu\left(h_\nu, \mathcal{R}\right)$ and $\gamma_\nu\left(h_\nu, \mathcal{R}\right)$ appearing in (34) can be simplified when $N = 2$ and can be related explicitly to the functions $\{w_n(\cdot)\}$, $\{\triangle w_n(\cdot)\}$, $\{h_m(\cdot)\}$, $\{\triangle h_m(\cdot)\}$ introduced in [1, Sec. II] to model a 2-D map $f(\mathbf{p})$.

Let us focus on the x coordinate ($\nu = x$); the integrals of $\lambda_x\left(h_x, \mathcal{R}\right)$ and $\gamma_x\left(h_x, \mathcal{R}\right)$ can be decomposed in x and y as

$$\lambda_x\left(h_x, \mathcal{R}\right) = \iint_{\mathcal{P}_\nu(h_\nu)} dx\, dy = \int_{y \in \mathcal{Y}} \int_{x \in \mathcal{P}_\nu(h_\nu, y)} dx\, dy \tag{35}$$

$$\gamma_x\left(h_x, \mathcal{R}\right) = \iint_{\tilde{\mathcal{P}}_\nu(h_\nu)} dx\, dy = \int_{y \in \mathcal{Y}} \int_{x \in \tilde{\mathcal{P}}_\nu(h_\nu, y)} dx\, dy \tag{36}$$

where the sets

$$\mathcal{P}_x\left(h_x, y\right) \triangleq \{x : f\left(x, y\right) > 0 \wedge f\left(x + h_x, y\right) > 0\} \subset \mathbb{R}$$

and

$$\tilde{\mathcal{P}}_x\left(h_x, y\right) \triangleq \{x : f\left(x, y\right) > 0 \wedge f\left(x + h_x, y\right) \wedge f\left(x - h_x, y\right) > 0\} \subset \mathbb{R}$$

represent slices of $\mathcal{P}_x\left(h_x\right)$ and $\tilde{\mathcal{P}}_x\left(h_x\right)$, respectively. These sliced sets are composed by 1-D segments and effectively represent 1-D maps as those described in Sec. 2.2 and Sec. 3.2. More precisely, $\mathcal{P}_x\left(h_x, y\right)$ is composed by $N_r = N_h(y)$ segments having widths $\{w_n(y)\}$ and centred around the points $\{c_{x,n}(y)\}$, where $n \in \mathcal{N}_h^o(y)$.

For these reasons $\mathcal{P}_x\left(h_x, y\right)$ has the same form of a slice of the $\mathcal{Q}$ set (depicted in Fig. 1 for a specific example 1-D map) and used in deriving the EZZB (see Sec. 3.2), obtained setting $t = x$, $u = h_x$. We note that:

1. each n-th segment of $\mathcal{P}_x\left(h_x, y\right)$, with $n \in \mathcal{N}_h^o(y)$, has width $w_n(y)$ and describes a triangle on the (h_x, x) plane (similar to $\mathcal{Q}_n$), which contributes to the inner integral of (35) a term $\omega\left(w_n(y), h_x\right)$ and to the inner integral of (36) a term $2\omega\left(\frac{w_n(y)}{2}, h_x\right)$, where the function $\omega\left(w, h_x\right)$ is defined as

$$\omega\left(w, h_x\right) \triangleq \left(w - h_x\right) \mathbb{I}_{[0;w]}\left(h_x\right) \tag{37}$$

2. the n-th segment of $\mathcal{P}_x(h_x, y)$, with $n \in \mathcal{N}_h^o(y)$, does overlap for some values of h_x with all previous segments, generating parallelograms on the (h_x, x) plane (similar to $\mathcal{Q}_{ov,n}$); for simplicity we consider only the overlap of the n-th segment with the the $(n-1)$-th segment, as done in the EZZB derivation. Each of such overlap contributes to the inner integral of (35) a term $\omega_{ov}(\triangle w_n(y), w_{n-1}(y), w_{n+1}(y), h_x)$, where $\omega_{ov}(\triangle w, w_1, w_2, h)$ is defined as

$$\omega_{ov}(\triangle w, w_1, w_2, h) \triangleq \begin{cases} h - \triangle w & h \in [\triangle w; \triangle w + w_2] \\ w_2 & h \in [\triangle w + w_2; \triangle w + w_1] \\ w_1 + w_2 + \triangle w - h & h \in [\triangle w + w_1; \triangle w + w_1 + w_2] \end{cases} \quad (38)$$

in the case $w_1 \geq w_2$; in the case $w_1 < w_2$, the function $\omega_{ov}(\triangle w, w_1, w_2, h)$ has the same form with w_1 and w_2 swapped.

For these reasons, for each value of the coordinate y, the inner integral of (35) can be written as

$$\sum_{n\in\mathcal{N}_h^o(y)} \omega(w_n(y), h_x) + \sum_{n\in\mathcal{N}_h^e(y)} \omega_{ov}(\triangle w_n(y), w_{n-1}(y), w_{n+1}(y), h_x) \quad (39)$$

whereas the inner integral of (36) can be written as

$$2 \sum_{n\in\mathcal{N}_h^o(y)} \omega\left(\frac{1}{2} w_n(y), h_x\right) \quad (40)$$

To obtain the expressions of $\lambda_x(h_x, \mathcal{R})$ and $\gamma_x(h_x, \mathcal{R})$, the outer integral on variable y has to be included:

$$\lambda_x(h_x, \mathcal{R}) = \int_{\mathcal{Y}} \sum_{n\in\mathcal{N}_h^o(y)} \omega(w_n(y), h_x)\, dy + \int_{\mathcal{Y}} \sum_{n\in\mathcal{N}_h^e(y)} \omega_{ov}(\triangle w_n(y), w_{n-1}(y), w_{n+1}(y), h_x)\, dy \quad (41)$$

$$\gamma_x(h_x, \mathcal{R}) = 2 \int_{\mathcal{Y}} \sum_{n\in\mathcal{N}_h^o(y)} \omega\left(\frac{1}{2} w_n(y), h_x\right) dy \quad (42)$$

Note that (41) and (42) coincide with [1, Eq. (23)] and [1, Eq. (24)], respectively.

References

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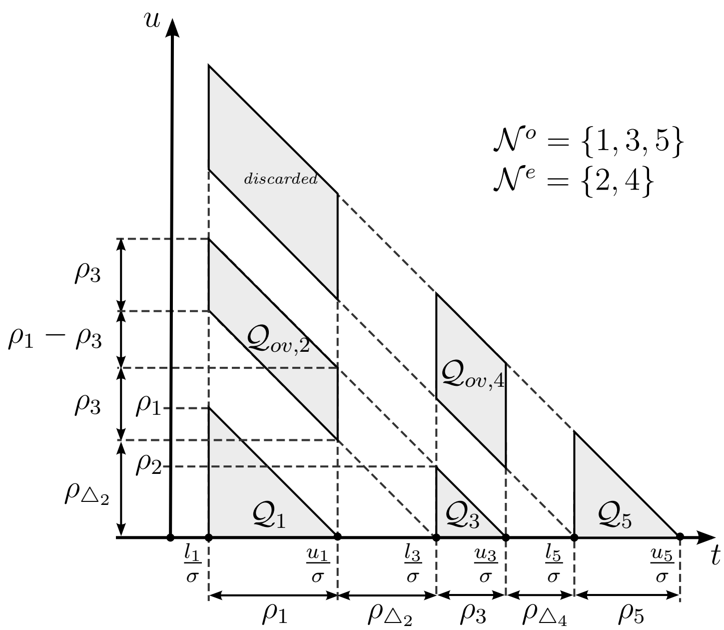

Figure 1: Representation of the set $\mathcal{Q}$ and of its subsets $\{\mathcal{Q}_i\}_{i\in\mathcal{N}^o}$ and $\{\mathcal{Q}_{ov,i}\}_{i\in\mathcal{N}^e}$ in the (t,u) plane ($N_r = 3$ is assumed). Note that the contribution from the "discarded" parallelogram is neglected in the evaluation of both the EZZB and the WWB.

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